Cosmology

SOM group meetings

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1 Introduction to Cosmology - David Albandea

References - Disclaimer

The notes of this section are essentially a summary of chapters 2 and 3 of Baumann's *Cosmology*, from which I took the notation, most of the reasonings (some of them literally) and some images. Dodelson's *Modern cosmology* was also used.

1.1 History of the Universe

Event	temperature	energy	time
Inflation	$< 10^{28}$ K (?)	$< 10^{16}$ GeV (?)	$> 10^{-34}$ s (?)
Dark matter decouples	?	?	?
Baryogenesis	?	?	?
EW phase transition	10^{15} K	100 GeV	10^{-11} s
Hadrons form (QCD PT)	10^{12} K	150 MeV	10^{-5} s
Neutrinos decouple	10^{10} K	1 MeV	1 s
Nuclei form (BBN)	10^{9} K	100 keV	200 s
Atoms form	3400 K	0.30 eV	260 000 yrs
Photons decouple (CMB)	2900 K	0.25 eV	380 000 yrs
First stars	50 K	4 meV	100 million yrs
First galaxies	20 K	1.7 meV	1 billion yrs
Dark energy dominates	3.8 K	0.33 meV	9 billion yrs
Einstein born	2.7 K	0.24 meV	13.8 billion yrs

Above we have a list of important events in the history of the universe. Some of them are already facts (BBN, recombination, CMB), and others are extrapolations made with extremely high confidence (EW and QCD phase transitions). Some others like dark matter production we know must have occurred, but their details are not very known.

We want to build the formalism to describe all of these processes in an expanding universe.

1.2 The expanding universe

1.2.1 Geometry

1. FLRW metric

   The geometry of the universe is characterized by the spacetime metric. In Minkowski space we have the line element

   $$ds^2 = -c^2 dt^2 + \delta_{ij} dx^i dx^j \tag{1}$$

   and therefore the metric is

   $$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{2}$$

   However, in GR the metric generally depends on the position in spacetime, $g_{\mu\nu}(t, \vec{x})$, so the curvature of spacetime is nontrivial. This curvature incorporates the effects of gravity.

   From measurements like the CMB we know that the universe is homogeneous and isotropic, so it can be represented by a time-ordered sequence of three-dimensional spatial slices Σ_t, so the four-dimensional line element can be written as

   $$ds^2 = -c^2 dt^2 + a^2(t) dl^2 \tag{3}$$

   with $dl^2 \equiv \gamma_{ij}(x^k) dx^i dx^j$ the line element on Σ_t and $a(t)$ the scale factor, which describes the expansion of the universe. This defines the metric $g_{\mu\nu} = (-1, a^2\gamma_{ij})$.

   The fact that we have homogeneity and isotropy leaves us with 3 possibilities for the curvature of the spatial slices Σ_t: it can be zero, positive or negative.

   - Flat space, with line element $dl^2 = d\vec{x}^2 = \delta_{ij} dx^i dx^j$.
   - Spherical space, with $dl^2 = d\vec{x}^2 + du^2$ and $\vec{x}^2 + u^2 = R_0^2$.
   - Hyperbolic space, with $dl^2 = d\vec{x}^2 - du^2$ and $\vec{x}^2 - u^2 = -R_0^2$.

   All cases can be compactly written as

   $$dl^2 = d\vec{x}^2 + k\frac{(\vec{x} \cdot d\vec{x})^2}{R_0^2 - k\vec{x}^2}, \qquad \text{for } k \equiv \begin{cases} 0 & E^3 \\ +1 & S^3 \\ -1 & H^3 \end{cases} \tag{4}$$

with E, S and H for Euclidean, Spherical and Hyperbolic spaces, respectively. Going to spherical coordinates the line element becomes

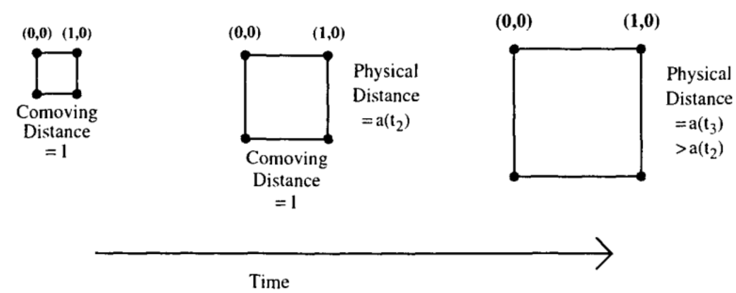
$$dl^2 = \frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\Omega^2, \quad d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2 \tag{5}$$

Substituting this into Eq. 3 we get the **Friedmann-Robertson-Walker (FRW) metric** in polar coordinates,

$$\boxed{ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\Omega^2 \right]} \tag{6}$$

Note that:

- The only independent components of the spacetime metric $g_{\mu\nu}$ are the scale factor $a(t)$ and the curvature scale R_0.
- The coordinate r is called a comoving coordinate, which is not a physical observable. The physical coordinate is $r_{\rm phys} = a(t)r$, that is, r in units of the scale factor $a(t)$.



- The line element ds^2 has a rescaling symmetry under which spacetime stays the same

$$a \to \lambda a, \quad r \to r/\lambda, \quad R_0 \to R_0/\lambda \tag{7}$$

so one uses this freedom to set the scale factor today, $t = t_0$, to be unity, $a(t_0) \equiv 1$. Then R_0 is the physical curvature scale today.

2. The Hubble Parameter

A galaxy with a trajectory $\vec{r}(t)$ in comoving coordinates and $\vec{r}_{\rm phys} = a(t)\vec{r}$ has physical velocity

$$\vec{v}_{\rm phys} \equiv \frac{d\vec{r}_{\rm phys}}{dt} = \frac{\dot{a}}{a}\vec{r}_{\rm phys} + a(t)\frac{d\vec{r}}{dt} \equiv H\vec{r}_{\rm phys} + \vec{v}_{\rm pec} \tag{8}$$

with $\vec{v}_{\rm pec} = a(t)d\vec{r}/dt$ the peculiar velocity (a velocity in units of the scale factor, i.e. a velocity measured by a comoving observer who follows the Hubble flow $H\vec{r}_{\rm phys}$); and H the Hubble parameter

$$\boxed{H \equiv \frac{\dot{a}}{a}} \tag{9}$$

Note that:

- If $\dot{a} = 0$ (universe does not expand) then the velocity of the object is just $\vec{v}_{\rm pec} = a(t)d\vec{r}/dt$.
- If $\dot{a} \neq 0$ then $H \neq 0$ and the term $H\vec{r}_{\rm phys}$ contributes. Note that H has units of $[T^{-1}]$, so $H\vec{r}_{\rm phys}$ is a velocity: it is the velocity at which space itself is expanding at $\vec{r}_{\rm phys}$.

1.2.2 Kinematics

We would like to know how free particles evolve in the FLRW metric. Free massive particles in curved spacetime follow **geodesics**, which is the timelike curve $x^\mu(\tau)$ which extremizes the action, and this extremal path satisfies

$$\boxed{\frac{dx^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}\,, \quad \text{with} \quad \Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \tag{10}$$

$\Gamma^\mu_{\alpha\beta}$ are the so-called Christoffel symbols. In terms of the momentum $P^\mu \equiv m\frac{dx^\mu}{d\tau}$, we can rewrite the geodesic equation as

$$P^\alpha\left(\partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta}P^\beta\right) = 0 \quad \Longleftrightarrow \quad P^\alpha\nabla_\alpha P^\mu = 0 \tag{11}$$

where the quantity in brackets is the so-called **covariant derivative** of the four-vector P^μ. In general in curved spacetime, derivatives need to be generalized to covariant derivatives,

$$\partial_\alpha A^\mu \to \nabla_\alpha A^\mu \equiv \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\beta}A^\beta \tag{12}$$

Note that the Christoffel symbols can be computed given a metric, so one can evaluate them for the FLRW metric in 6, $g_{\mu\nu} = (-1, a^2\gamma_{ij})$. Considering the $\mu = 0$ component of eq. 11 one finds

$$\frac{E}{c^3}\frac{dE}{dt} = -\frac{1}{c}a\dot{a}\gamma_{ij}P^iP^j \tag{13}$$

For a **massless particle** with $g_{\mu\nu}P^\mu P^\nu = -c^{-2}E^2 + a^2\gamma_{ij}P^iP^j = 0$ this implies

$$\frac{1}{E}\frac{dE}{dt} = -\frac{\dot{a}}{a} \Longleftrightarrow E \propto a^{-1} \tag{14}$$

which indicates that the wavelength of radiation gets **redshifted** over time due to the expansion of the universe.

Analogously, for a massive particle with $g_{\mu\nu}P^\mu P^\nu = -m^2c^2$ one finds that the *physical* peculiar three-momentum $p^2 \equiv g_{ij}P^iP^j$ scales as $p \propto a^{-1}$, meaning that freely-falling particles eventually converge onto the Hubble flow (see eq. 8).

1.2.3 Dynamics

We would like to know how the expansion of the universe evolves as a function of time. This is given by the scale factor $a(t)$ and its evolution can be obtained from the Einstein equation

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}, \tag{15}$$

with $G_{\mu\nu}$ the Einstein tensor (a measure of the "spacetime curvature" of the universe) and $T_{\mu\nu}$ the momentum tensor (a measure of the "matter content" of the universe). We need to determine both to solve for the evolution of $a(t)$ as a function of the matter content.

The Einstein tensor depends on the Christoffel symbols and can be computed for a given metric,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{16}$$

with $R_{\mu\nu}$ the Ricci tensor and $R = R^\mu{}_\mu$. The Ricci tensor is defined as

$$R_{\mu\nu} \equiv \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}\,. \tag{17}$$

On the other hand, the energy-momentum tensor is required to be a **perfect fluid** to be consistent with the universe being homogeneous and isotropic. The physical meaning of the components are

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0j} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{energy flux} & \text{stress tensor} \end{array}\right) \tag{18}$$

Isotropy implies that the off-diagonal elements must vanish, and homogeneity implies that the pressure must be the same in all directions, so we have

$$T^{\mu}{}_{\nu} = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{19}$$

where the energy density $\rho(t)$ and the preassure $P(t)$ are functions of time. For a general observer

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P g_{\mu\nu}} \tag{20}$$

where $U_\mu \equiv dx^\mu/d\tau$ is the four-velocity between the particles and the observer.

1. Continuity equation

   We want to know how the density and pressure evolve with time. As a simpler example, the number density N^μ in Minkowski space needs to satisfy the continuity equation

   $$\partial_0 N^0 = -\partial_i N^i \iff \partial_\mu N^\mu = 0 \tag{21}$$

   which means that, if the number of particles is conserved, the rate of change of the number density N^0 must equal the divergence of the flux of the particles N^i. In curved spacetime the equation is generalized with the covariant derivative

   $$\nabla_\mu N^\mu = 0 \tag{22}$$

   Analogously, the continuity equation for the energy density, $\dot{\rho} = -\partial_i \pi^i$, and the Euler equation for the evolution of the momentum density, $\dot{\pi}_i = \partial_i P$, are combined into a four-component conservation equation for the energy-momentum tensor,

   $$\boxed{\nabla_\mu T^{\mu}{}_{\nu} = 0} \tag{23}$$

   Using the FLRW metric and eq. 20 in eq. 23 for the $\nu = 0$ component one gets the evolution of the energy density,

   $$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0}, \tag{24}$$

which describes the "energy conservation" in the cosmological context. Since the usual notion of energy conservation in flat space relies on a symmetry under time translations and we don't have that due to the expanding space, the standard notion of energy conservation is replaced by this equation.

Most cosmological fluids can be parametrized in terms of a constant **equation of state** relating pressure and density,

$$P = w\,\rho c^2 \tag{25}$$

with w a constant. Plugging this back into eq. 24 one finds how the energy density dilutes with the scaling factor $a(t)$,

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \implies \boxed{\rho \propto a^{-3(1+w)}} \tag{26}$$

2. Matter, radiation and dark energy

   The dilution of the energy density of eq. 26 depends on w, which depends on the kind of fluid we are studying.

   - **Matter** is a fluid whose pressure is much smaller than its energy density, $|P| \ll \rho c^2$, which means $w = 0$. From eq. 26 this implies that the energy density scales as

     $$\rho_m \propto a^{-3} \tag{27}$$

     That is, the energy in a region stays constant, but the region of space increases as $V \propto a^3$. Examples of matter are baryons and dark matter.

   - **Radiation** is anything for which the pressure is one third of the energy density, $P = \frac{1}{3}\rho c^2$, so that $w = 1/3$. This implies

     $$\rho_r \propto a^{-4}, \tag{28}$$

     which includes the space expansion, $V \propto a^3$, and the redshifting of the energy of the particles that we found in eq. 14, $E \propto a^{-1}$. Examples of radiation are light particles, photons and neutrinos.

   - The universe today is dominated by a mysterious form of **dark energy** with negative pressure, $P = -\rho c^2$, and hence $w = -1$.

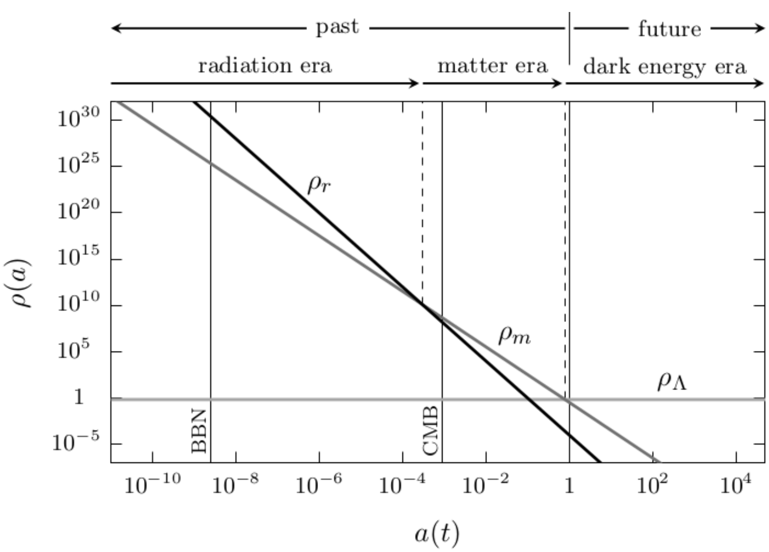
This means the the energy density remains constant with the expansion of the universe

$$\rho \propto a^0 \tag{29}$$

Since energy density doesn't dilute, energy has to be created as the universe expands; but this doesn't violate the conservation of energy as long as the continuity equation in 24 is satisfied. This dark energy could be accounted for if we added a cosmological constant to the Einstein equation that does not change the conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$, that is,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{30}$$

The different scalings of the energy density depending on the kind of fluid indicates that in the past there were eras in which only one component dominated, as seen in the figure below.



This fact can simplify the study of a particular era, since then only the dominant fluid has to be considered for the evolution of $a(t)$.

3. Friedmann equations

Now we can evaluate the Einstein equation 15. The $\mu = 0$, $\nu = 0$ component yields the **Friedmann equation**, which is the fundamental equation describing the evolution of the scale factor,

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2}} \Leftrightarrow \boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2}} \tag{31}$$

where ρ should be understood as the sum of all contributions to the energy density in the universe.

To write the Friedmann equation as a closed form equation for $a(t)$ we have to specify the evolution of the density, $\rho(a)$, as in the continuity equation, eq. 26.

It is common to rewrite the Friedmann equation as a function of relative densities with respect to the critical density

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \tag{32}$$

which is the density needed to have a flat universe with $k = 0$. The subcript "0" means today, at $t = t_0$. The relative densities are then defined as

$$\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{\text{crit},0}}, \quad i = r, m, \Lambda \ldots \tag{33}$$

with r for radiation, m for matter, Λ for the cosmological constant. The subscript "0" is usually dropped. The Friedmann equation, eq. 31, can be then rewritten as

$$\boxed{\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda} \tag{34}$$

with the curvature "density" parameter $\Omega_k \equiv -kc^2/(R_0 H_0)$. At $t = t_0$ this yields

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k \equiv \Omega_0 + \Omega_k \tag{35}$$

with Ω_0 the sum of all matter components.

A central task in cosmology is to measure the parameters occuring in the Friedmann equation in eq. 34 and hence determine the composition of the universe.

Many of these components have been measured from the CMB: the COBE and Planck satellites measured the temperature of the universe at the time of the photons decoupling, and also the temperature fluctuations.

To be able to use these results we need to know the dependence of the energy density with temperature, so first we need to study thermal equilibrium in an expanding universe.

(a) Newtonian derivation of the Friedmann equation

Some intuiton of the form of the Friedmann equation can be developted from a nonrelativitsic Newtonian analysis. Consider an expanding sphere of matter of uniform mass density $\rho(t)$ and radius $R(t) = a(t)R_0$, and let's study the dynamics of a test particle on the surface of the sphere. The acceleration of the particle is

$$\ddot{R} = -\frac{GM(R)}{R^2}, \quad M(R) = \frac{4\pi}{3}R^3\rho. \tag{36}$$

Note that the mass enclosed in the sphere $M(R)$ is constant. Multiplying this by $\dot{R}$,

$$\dot{R}\ddot{R} = -GM(R)\frac{\dot{R}}{R^2} \quad \Longrightarrow \quad \frac{d}{dt}(\dot{R})^2 = GM(R)\frac{d}{dt}\left(\frac{1}{R}\right) \tag{37}$$

and integrating, we get

$$\frac{1}{2}\dot{R}^2 - \frac{GM(R)}{R} = E \tag{38}$$

where the first term is a kinetic term per unit mass and the second term a potential term per unit mass; therefore, the integration constant E is the energy per unit mass of the particle. In terms of the scale factor and the density, this becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{a^2R_0^2} \tag{39}$$

which is the form of the Friedmann equation of eq. 31 if we identify $2E$ with $-kc^2$.

This example describes the evolution of a particle in a sphere of mass $M(R)$. However, the Friedmann equations above describe the evolution of spacetime itself: the spatial curvature would then be related to the total energy of the spacetime region, with flatness arising if the kinetic and potential energies precisely add up to zero.

(b) Complete Friedmann equation

Adding the cosmological constant, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2R_0^2} + \frac{\Lambda c^2}{3} \tag{40}$$

where ρ is the total energy density of the universe.

1.3 The Hot Big Bang and thermal equilibrium

The blackbody spectrum of the CMB indicates that the early universe was in thermal equilibrium, consisting of a hot gas of weakly interacting particles. One cannot describe the hot gas by the positions and velocities of each particle, so we will need to characterize the properties of the gas statistically.

1.3.1 Density of states

A key concept in statistical mechanics is the probability that a particle chosen at random has momentum $\vec{p}$. The probability distribution function $f(\vec{p}, t)$ can be very complicated, but if we wait long enough compared to the typical interaction timescale the system will reach *equilibrium* and is characterized by a time-independent distribution function, where the gas has reached a state of maxium entropy:

$$f(p, T) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} \tag{41}$$

Here the $+$ sign is for fermions and the $-$ sign for bosons.

In a region of "volume" d^3xd^3p there can be a number of phase space elements $d^3xd^3p/(2\pi\hbar)^3$, since by Heisenberg's principle no particle can be localized into a region of phase space smaller than $(2\pi\hbar)^3$. Then, the average number of particles per unit of phase space can be seen as

$$\frac{dN}{d^3xd^3p} = \frac{g}{(2\pi)^3} f(p, T) \tag{42}$$

with N the total number of particles, g the number of degrees of freedom of the particle species in particular and we have set $\hbar = 1$. Integrating over p gives the number density $n(T)$,

$$\boxed{n(T) = \frac{g}{(2\pi)^3} \int d^3p\, f(p, T)} \tag{43}$$

If we didn't have $f(p,T)$ the result would be $g\frac{V_{ph}}{(2\pi)^3}$, i.e. the total phase space volume V_{ph} divided by the volume of one state, which yields the total number of states in phase space. Therefore, this integral gives the weighted average value of states of the particle, per volume (it's a density).

Also, the energy density and preassure of the gas are given by

$$\boxed{\rho(T) = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)E(p)}, \tag{44}$$

$$\boxed{P(T) = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)\frac{p^2}{3E(p)}} \tag{45}$$

Each particle species has its own distribution function f_i, and therefore its own density and pressure n_i, ρ_i and P_i. Species that are in thermal equilibrium share a common temperature, $T_i = T$, so their densities and preassures can then only differ because of differentces in their masses and chemical pontentials.

1.3.2 The Primordial Plasma

We want to relate the densities and pressures of the different species in the primordial plasma to the overall temperature of the universe.

Setting the chemical potential to zero and using $E(p) = \sqrt{p^2+m^2}$ one gets

$$n = \frac{g}{2\pi^2}\int_0^\infty dp\,\frac{p^2}{\exp\left[\sqrt{p^2+m^2}/T\right]\pm 1} \tag{46}$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty dp\,\frac{p^2\sqrt{p^2+m^2}}{\exp\left[\sqrt{p^2+m^2}/T\right]\pm 1} \tag{47}$$

and defining the dimensionless variables $x \equiv m/T$ and $\xi \equiv p/T$,

$$n = \frac{g}{2\pi^2}T^3 I_\pm(x), \qquad I_\pm(x) \equiv \int_0^\infty d\xi\,\frac{\xi^2}{\exp\left[\sqrt{\xi^2+x^2}\right]\pm 1} \tag{48}$$

$$\rho = \frac{g}{2\pi^2}T^4 J_\pm(x), \qquad J_\pm(x) \equiv \int_0^\infty d\xi\,\frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left[\sqrt{\xi^2+x^2}\right]\pm 1} \tag{49}$$

Where the functions $I_\pm(x)$ and $J_\pm(x)$ need to be evaluated numerically in general. But can be determined analytically in the relativistic and non-relativistic limits.

1. Relativistic limit

   Taking the limit $x \to 0$ the integral reduces to

$$I_{\pm}(0) = \int_0^{\infty} d\xi \, \frac{\xi^2}{e^{\xi} \pm 1}. \tag{50}$$

   Taylor expanding the integrand one can find

$$I_{+}(0) = \frac{3}{4} I_{-}(0), \quad I_{-} = 2\zeta(3) \tag{51}$$

   and therefore

$$\boxed{n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}} \tag{52}$$

   One can also show that

$$J_{+}(0) = \frac{7}{8} J_{-}(0), \quad J_{-}(0) = 6\zeta(4) \tag{53}$$

   and therefore

$$\boxed{\rho = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}} \tag{54}$$

   So given the temperature of a relativistic particle species we also know its number and energy density.

2. Non-relativistic limit

   Taking the limit $x \gg 1$ for tempratures below the particle mass the integral becomes the same for fermions and bosons,

$$I_{\pm}(x) \approx \int_0^{\infty} d\xi \, \frac{\xi^2}{e^{\sqrt{\xi^2 + x^2}}} \tag{55}$$

   One can Taylor expand the square root to first order and perform the remaining Gaussian integral to find

$$I_{\pm}(x) = \sqrt{\frac{\pi}{2}} x^{3/2} e^{-x} \tag{56}$$

and comparing with the relativistic case $x = 0$,

$$\frac{I_\pm(x)}{I_-(0)} \approx 0.5 x^{3/2} e^{-x} \ll 1 \tag{57}$$

meaning that massive particles are exponentially rare at low temperatures.

The density of the non-relativistic gas as a function of temperature is

$$\boxed{n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}} \tag{58}$$

The energy density assuming $E(p) = \sqrt{m^2 + p^2} \approx m + p^2/2m$ is

$$\rho \approx mn + \frac{3}{2} nT \tag{59}$$

and one can show that the pressure is

$$P = nT \tag{60}$$

which is nothing but the ideal gas law, $PV = Nk_BT$. Since $T \ll m$, we have $P \ll \rho$, so that the gas acts like pressureless dust ("matter").

By comparing the relativistic and non-relativistic limit one sees that the number density, energy density and pressure of a particle species fall exponentially (are "Boltzmann suppressed") as the temperature drops below the mass of the particles. This can be interpreted as the annihilation of particles and anti-particles.

1.3.3 Energy density as a function of temperature

The total energy density ρ of the early universe is the sum over all contributions of species

$$\rho = \sum_i \frac{g_i}{2\pi^2} T_i^4 J_\pm(x_i) \tag{61}$$

where the contribution of species i was derived in eq. 48. In general different species can have different temperatures T_i, but this is only relevant for neutrinos after electron-positron annihilation. It is common to define a "temperature of the universe" T so that

$$\boxed{\rho_{\text{universe}}(T) \equiv \frac{\pi^2}{30} g_*(T) T^4} \tag{62}$$

with the **effective number of degrees of freedom**

$$g_*(T) \equiv \sum_i g_i \left(\frac{T_i}{T}\right)^4 \frac{J_\pm(x_i)}{J_-(0)} \tag{63}$$

Since as we have seen the energy density of relativistic species is much greater than that of non-relativistic species, one usually only includes the relativistic species in g_* and, using $T_i \gg m_i \implies x \approx 0$ and the results of $J_\pm(0)$ of eq. 53, it reduces to

$$\boxed{g_*(T) \equiv \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4} \tag{64}$$

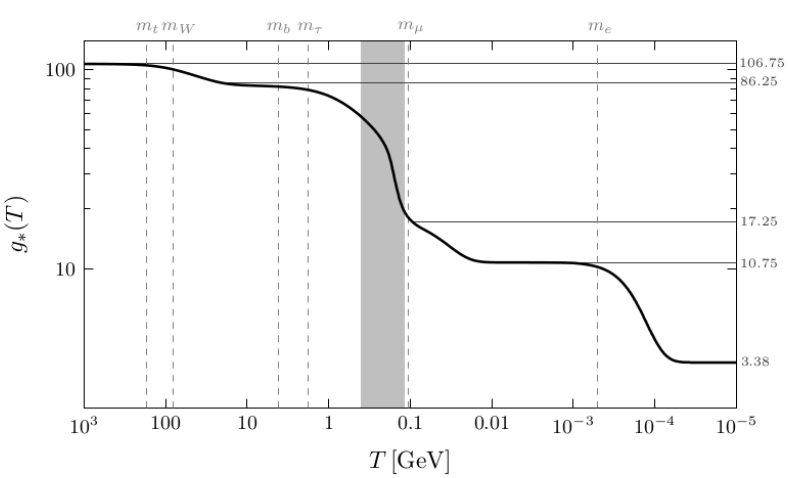
If all particles are in equilibrium at a common temperature T, determining $g_*(T)$ is simply summing over the different particles of the Standard Model which are relativistic at temperature T.

Considering a flat universe ($k = 0$) one can use the Friedmann equation of eq. 31 to relate the expansion history of the universe to its temperature in a radiation-dominated era,

$$\boxed{H^2 = \frac{\rho}{3M_{\text{Pl}}^2} \simeq \frac{\pi^2}{90} g_* \frac{T^4}{M_{\text{Pl}}^2}} \tag{65}$$

with $M_{\text{Pl}} \equiv \sqrt{\hbar c/(8\pi G)} = 2.4 \times 10^{18}$ GeV.

1. Evolution of $g_*(T)$



At $T \gtrsim 100$ GeV, all particles of the Standard Model were relativistic.

- $g_\gamma = 2$, for the two polarization transverse to the direction of propagation.
- A massive particle of spin s has $g = 2s + 1$ polarization states.
- $g_{W^\pm,Z} = 3$ and a total of 3×3 degrees of freedom.
- Gluons are massless, so $g_g = 2$, but there are 8 of them, so 8×2 dof.
- Fermions have $s = 1/2$ and contribute with 2 spin states.
- Charged leptons ($e^\pm$, $\mu^\pm$, $\tau^\pm$) contribute $3 \times 2 \times 2 = 12$.
- Quarks (t, b, c, s, d, u) have 3 different colors, so $6\times2\times3\times2 = 72$.
- Neutrinos, whether Dirac or Majorana, contribute with 1 internal degree of freedom.

Adding up the internal degrees of freedom,

$$g_b = 28, \quad g_f = 90 \qquad \Longrightarrow \qquad g_* = g_b + \frac{7}{8} g_f = 106.75 \tag{66}$$

when all species were relativistic with common temperature T. As temperature drops and particle species become non-relativistic, one estimates g_* by counting just the relativistic degrees of freedom with $m \ll T$ and discarding the rest.

- Top quarks annihilate first, so at $T \sim \frac{1}{6} m_t \sim 30$ GeV we have $g_* = 96.25$.
- Higghs boson and $W^\pm$ and Z annihilate next and at $T \sim 10$ GeV we have $g_* = 86.25$.
- The bottom quarks annihilate (75.75) followed by the tau leptons (61.75).
- Before strange quarks annihilate matter undergoes the QCD phase transition. At $T \sim 150$ MeV quarks combine into baryons and mesons: although there are many different species of them, all except the pions ($\pi^\pm$, π^0) are non-relativitstic below the temperature of the QCD phase transition and are Boltzmann suppressed, so we only have pions, electrons, muons, neutrinos and photons as relativistic. The three types of pions are spin-0 bosons, which carry a total of $g = 3$ internal degrees of freedom, so we have $g_* = 17.25$.

- After the QCD transition, the muons and pions annihilate leading to $g_* = 10.75$.
- Finally, electrons and positrons will annihilate.

1.3.4 Entropy and expansion history

To describe the evolution of the universe it is useful to track a conserved quantity, and **entropy** is more informative than energy. The total entropy of the universe only increases or stays constant (2nd law of thermodynamics). We will show that entropy is conserved in equilibrium.

1. Conservation of entropy

   Assuming chemical potentials small, the first law of thermodynamics reads

   $$TdS = dU + pdV \tag{67}$$

   defining the entropy density $s \equiv S/V$ and noting that s and ρ do not depend on the volume V, one can write

   $$(Ts - \rho - P)\, dV + V\left(T\frac{ds}{dT} - \frac{d\rho}{dT}\right) dT = 0 \tag{68}$$

   where the two brackets have to vanish separately for arbitrary variations dV and dT. The vanishing of the first bracket implies

   $$\boxed{s = \frac{\rho + P}{T}} \tag{69}$$

   and the second bracket enforces

   $$\frac{ds}{dT} = \frac{1}{T}\frac{d\rho}{dT} \quad \Longleftrightarrow \quad \frac{d(sa^3)}{dt} = 0 \tag{70}$$

   where we have used the continuity equation $d\rho/dt = -3H(\rho + P) = -3HTs$. This equation means that the total entropy is conserved in equilibrium and that the density evolves as $s \propto a^{-3}$.

   If we have chemical potential, we would have

   $$s = \frac{\rho + P - \mu n}{T}, \quad \frac{d(sa^3)}{dt} = -\frac{\mu}{T}\frac{d(na^3)}{ddt} \tag{71}$$

   so entropy is conserved either if the chemical potential is small, $\mu \ll T$, or if no particles are created or destroyed.

2. Relativistic species

   For a collection of different relativistic species, the total entropy density is

$$s = \sum_i \frac{\rho_i + P_i}{T_i} \equiv \frac{2\pi^2}{45} g_{*S}(T) T^3 \tag{72}$$

   where we have used $P_i = \frac{1}{3}\rho_i$ and the relativistic expression for the density in eq. 54 to define the "effective number of degrees of freedom in entropy" g_{*S}. If we are away from mass thresholds,

$$g_{*S}(T) \approx \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3 \tag{73}$$

   When all species are in equilibrium with $T_i = T$ then $g_{*S} = g_*$. In our universe this is the case until $t \approx 1$ s.

   A consequence of entropy conservation is that

$$g_{*S}(T) T^3 a^3 = \text{const} \quad \text{or} \quad T \propto g_{*S}^{-1/3} a^{-1} \tag{74}$$

   Away from mass thresholds, g_{*S} is approximately constant and the temperature scales as $T \propto a^{-1}$. But when a particle species becomes non-relativistic and disappears, its entropy is transferred to the other relativistic species still present in the thermal plasma, acusing T to decrease slightly more slowly than a^{-1}.

3. Temperature of neutrinos

   Neutrinos decoupled from the thermal plasma because its interaction became smaller then the Hubble rate,

$$\Gamma < H \tag{75}$$

   Shortly after neutrinos decoupled, the temperature dropped below the electron mass, so electrons and positrons became non-relativistic. In this process the photons in the plasma were "heated" due to entropy conservation (their temperature decreased more slowly, see eq. 74), but neutrinos were not because they already decoupled from the plasma.

   Before and after electron decoupling, we have

$$g_{*S} = \begin{cases} 2 + \frac{7}{8} \times 4 = \frac{11}{2} & T \gtrsim m_e \\ 2 & T < m_e \end{cases} \tag{76}$$

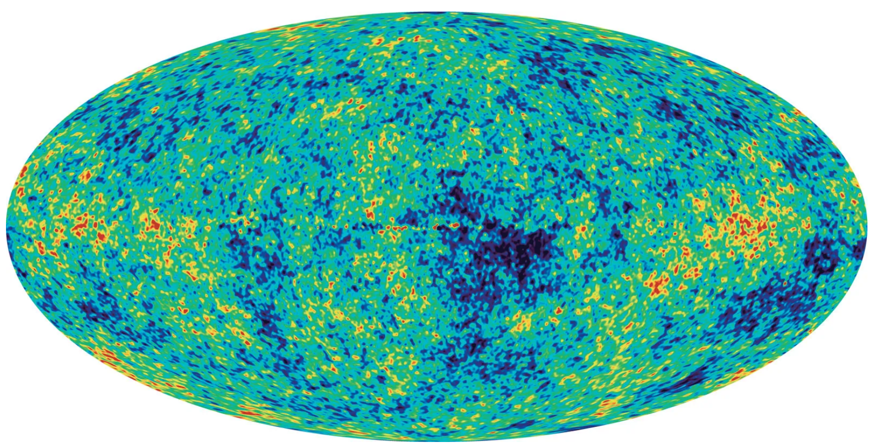
Enforcing $g_{*S}(aT_\gamma)^3$ to remain constant, aT_γ increases by a factor $(11/4)^{1/3}$ after electron-positron annihilation, while aT_ν remains constant. This means that the neutrino temperature is slightly lower than the photon temperature,

$$\boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma} \tag{77}$$

relation which holds until the present.

1.4 Measurements from our Universe

1.4.1 From CMB temperature: photon and neutrino relic densities



The COBE satellite found the tempreature of the CMB blackbody spectrum to be

$$T_0 = (2.7260 \pm 0.0013) \text{ K} \tag{78}$$

Using eqs. 52 and 54 we can relate this temperature to the number density and energy density of the relic photons,

$$n_{\gamma,0} = 0.24 \times \left(\frac{k_B T_0}{\hbar}\right)^3 \approx 410 \text{ photons cm}^{-3} \tag{79}$$

$$\rho_{\gamma,0} = 0.66 \times \frac{(k_B T_0)^4}{(\hbar c)^3} \approx 4.6 \times 10^{-34} \text{ g cm}^{-3} \tag{80}$$

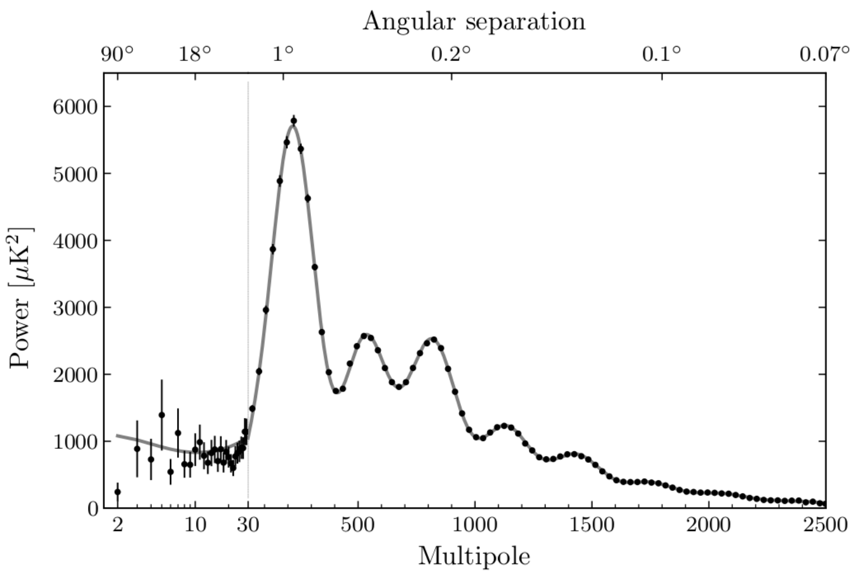
In terms of the critical density, the energy density of the photons is

$$\Omega_\gamma \approx 5.35 \times 10^{-5} \tag{81}$$

The universe is also filled with a background of relic neutrinos and their energy density is 68% of the relic photons (due to eq. 77), yielding a total radiation energy of

$$\Omega_r = 8.99 \times 10^{-5} \tag{82}$$

1.4.2 From CMB temperature fluctuations: curvature and DM relic densities



The COBE satellite also discovered that the CMB temperature varies with position on the sky with fluctuations $\Delta T/T \sim 10^{-5}$. The plot below shows the two-point correlation function of the CMB temperature fluctuations:

The positions of the peaks depend on the spatial curvature of the universe, which depend on the cosmological model. The measurements suggest an upper bound for any amount of spatial curvature

$$|\Omega_k| < 0.005 \tag{83}$$

which constrains the curvature to be less than 1%. Since the curvature contribution scales as a^{-2} while matter and radiation scale as a^{-3} and a^{-4}, it is believed that effects of curvature were completely negligible at earlier times.

The pattern of CMB fluctuations depends sensitively on the amount of dark matter, and the inferred dark matter density today is

$$\Omega_c \approx 0.27 \tag{84}$$

where the subscript (c) indicates that we are assuming a "cold" form of dark matter with equation of state $w_c \approx 0$.

1.4.3 From BBN and CMB: baryon number bayon relic densities

Since $s \propto a^{-3}$, the number of particles in a comoving volume is proportional to the number density n_i divided by the entropy density,

$$N_i \equiv \frac{n_i}{s} \tag{85}$$

If particles are not produced nor destroyed then $n_i \propto a^{-3}$ and N_i is a constant. An important example is the total baryon number after baryogenesis,

$$\frac{n_B}{s} \equiv \frac{n_b - n_{\bar{b}}}{s} \tag{86}$$

and the related baryon-to-photon ratio

$$\eta \equiv \frac{n_B}{n_\gamma} = 1.8 g_{*S} \frac{n_B}{s} \tag{87}$$

which after electron-positron annihilation becomes a conserved quantity, $\eta \approx 7 n_B/s$, and is therefore a useful measure of the baryon content of the universe.

Both Big Bang Nucleosynthesis (BBN) and the CMB observations show that baryons only make up 5% of the critical density,

$$\Omega_b \approx 0.05 \tag{88}$$

1.4.4 From supernovae luminosities: dark energy relic density

Assuming a flat universe (as suggested by the CMB observations), the data from distant supernovae luminosities can only be fit if the universe contains a significant amount of dark energy,

$$\Omega_\Lambda \approx 0.68. \tag{89}$$